W26. Let be the function $f:[0,1]\to\mathbb{R}$ integrable such that $f(1)=1$ and $\int_x^y f(t)dt=\frac{1}{2}(yf(y)-xf(x)),\forall x,y\in[0,1]$. Find

$$I=\int_0^{\pi/4} f(x)\cdot\tan^2 x dx$$

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Solution by Arkady Alt, **San Jose**, **California**, **USA**.

$\int_x^y f(t)dt=\frac{1}{2}(yf(y)-xf(x))=\int_x^y \frac{1}{2}(tf(t))'dt \Leftrightarrow \int_x^y\left(f(t)-\frac{1}{2}(tf(t))'\right)dt=0$

In particular $\int_0^x f(t)dt=\int_0^x \frac{1}{2}(tf(t))'dt$ for any $x\in(0,1]$. Then $\left(\int_0^x f(t)dt\right)_x'=$ $\left(\int_0^x \frac{1}{2}(tf(t))'dt\right)_x' \Leftrightarrow f(x)=\frac{1}{2}(xf(x))' \Leftrightarrow f(x)=\frac{1}{2}(f(x)+xf'(x)) \Leftrightarrow$ $f(x)=xf'(x)\Leftrightarrow \frac{xf'(x)-f(x)}{x^2}=0\Leftrightarrow\left(\frac{f(x)}{x}\right)'=0\Leftrightarrow\frac{f(x)}{x}=c.$

Since $f(1)=1$ then $f(x)=x$ and, therefore,

$$I=\int_0^{\pi/4} x\tan^2 x dx=\left[\begin{array}{c}u'=\tan^2 x; u=\tan x-x\\ v=x; v'=1\end{array}\right]=$$

$$((x\tan x-x^2))_0^{\pi/4}-\int_0^{\pi/4}(\tan x-x)dx=\left(\frac{\pi}{4}-\left(\frac{\pi}{4}\right)^2\right)+\left(\ln(\cos x)+\frac{1}{2}x^2\right)_0^{\pi/4}=$$

$$\frac{\pi}{4}-\left(\frac{\pi}{4}\right)^2-\frac{1}{2}\ln 2+\frac{1}{2}\cdot\left(\frac{\pi}{4}\right)^2=\frac{1}{32}(8\pi-\pi^2-16\ln 2).$$